week 12 disc 104

Wednesday, April 14, 2021 10:49 AM

5.9 Problems

Problem 1. *Use the Runge-Kutta method for systems to approximate the solutions of the following system of first-order differential equations and compare the results to the actual solutions.*

$u_1' = -4u_1 - 2u_2 + \cos(t) + 4\sin(t)$, $u_2' = 3u_1 + u_2 - 3\sin(t)$, $u_1(0) = 0$, $u_2(0) = -1$, $0 \le t \le 2$, $h = .1$.
Actual solutions $u_1(t) = 2e^{-t} - 2e^{-2t} + \sin(t)$ *and* $u_2(t) = -3e^{-t} + 2e^{-2t}$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}' = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} \cos(t) + 4\sin(t) \\ -3\sin(t) \end{pmatrix} \qquad \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

5.10 Problems

Problem 2. *Show that if the initial value problem* $y' = f(t,y)$, $a \le t \le b$, $y(a) = \alpha$ *is approximated by a one-step difference method:* $w_0 = \alpha$, $w_{i+1} = w_i + h\varphi(t_i, w_i, h)$. *And there exists* $h_0 > 0$ *and* $\varphi(t,w,h)$ *continuous and Lipschitz in the* w *variable with Lipschitz constant* L *on*

$$D = \{(t,w,h) | a \le t \le b, -\infty < w < \infty, 0 \le h \le h_0\}$$

then there exists a constant $K > 0$ *such that*

$$|u_i - v_i| \le K|u_0 - v_0|$$

for each $1 \le i \le N$ *whenever* $\{u_i\}_{i=1}^N$ *and* $\{v_i\}_{i=1}^N$ *satisfy the difference equation* $w_{i+1} = w_i + h\varphi(t_i, w_i, h)$

32. The Runge-Kutta method of order four can be written in the form

$$w_0 = \alpha,$$

$$\begin{aligned} w_{i+1} = w_i &+ \frac{h}{6} f(t_i, w_i) + \frac{h}{3} f(t_i + \alpha_1 h, w_i + \delta_1 h f(t_i, w_i)) \\ &+ \frac{h}{3} f(t_i + \alpha_2 h, w_i + \delta_2 h f(t_i + \gamma_2 h, w_i + \gamma_3 h f(t_i, w_i))) \\ &+ \frac{h}{6} f(t_i + \alpha_3 h, w_i + \delta_3 h f(t_i + \gamma_4 h, w_i + \gamma_5 h f(t_i + \gamma_6 h, w_i + \gamma_7 h f(t_i, w_i)))). \end{aligned}$$

Find the values of the constants

$$\alpha_1, \alpha_2, \alpha_3, \delta_1, \delta_2, \delta_3, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \text{ and } \gamma_7.$$

$$|u_1 - v_1| = |u_0 + h\varphi(t_0, u_0, h) - v_0 - h\varphi(t_0, v_0, h)|$$

$$\le |u_0 - v_0| + h\,\underbrace{|\varphi(t_0, u_0, h) - \varphi(t_0, v_0, h)|}_{\le |u_0 - v_0| L}$$

$$\le |u_0 - v_0|(1 + hL)$$

$$|u_2 - v_2| \le \underset{\text{reason}}{\overset{\text{same}}{\cdots}} \le \underbrace{|u_1 - v_1|}(1 + hL) \le |u_0 - v_0|(1 + hL)^2$$

By induction

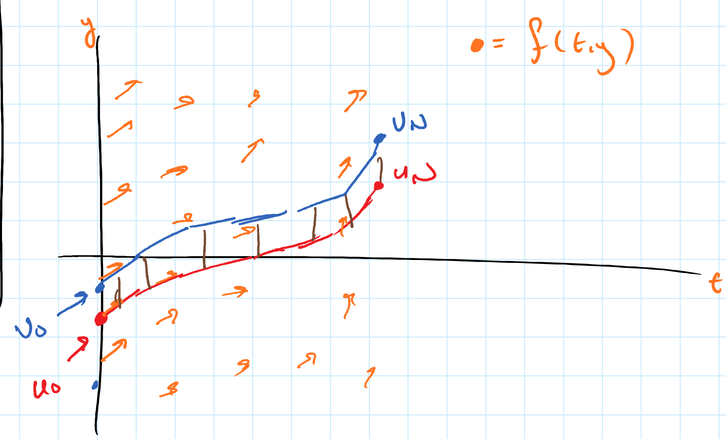
$$\underline{|u_N - v_N|} \le |u_0 - v_0|(1 + hL)^N$$

$$\Rightarrow \forall\ j = 1, \dots, N$$

$$|u_j - v_j| \le |u_0 - v_0| \underbrace{(1 + hL)^N}_{K}$$

$$\lim_{N \to \infty}\left(1 + \frac{\cancel{h}L(b-a)}{N}\right)^N = e^{L(b-a)}$$

What does this mean?

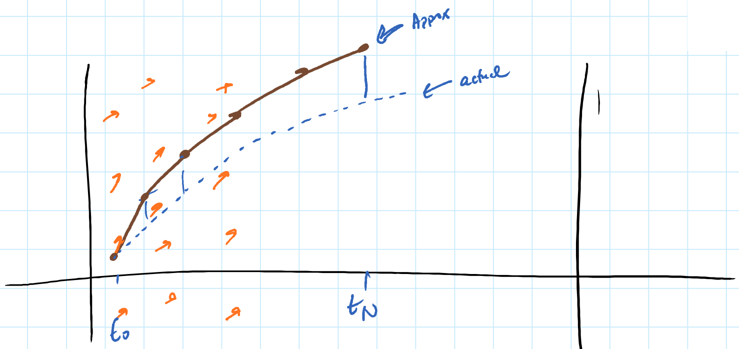


Problem 4. *Given the multistep method*

$$w_{i+1} = -\frac{3}{2}w_i + 3w_{i-1} - \frac{1}{2}w_{i-2} + 3hf(t_i, w_i)$$

for $i = 2, \ldots, N-1$, with starting values w_0, w_1, w_2. (a) Find the local truncation error. (b) comment on consistency, stability, and convergence.

Problem 3. *Show that Runge-Kutta method of order four is consistent using the results of exercise 32 in section 5.4.*



$\left[\boxed{|y_{i+1} - w_{i+1}|} \right]$ Computed assuming correct at $i=1$

$y'(t) = f(t, y(t))$

y_i = actual solution at t_i

4) $\tau_{i+1}(h)$ = "Local truncation error" = $\dfrac{y_{i+1} - (-\frac{3}{2}y_i + 3y_{i-1} - \frac{1}{2}y_{i-2})}{h} - 3f(t_i, y_i)$

$|\tau_{i+1}(h)|$

$$\frac{y(t+h) + \frac{3}{2}y(t_i) - 3y(t_i - h) - \frac{1}{2}y(t_i - 2h)}{h} - 3f(t_i, y(t_i))$$

Let $P(t)$ be a 3^{rd} order poly interpolating $y(t)$ at $t_{i-2}, t_{i-1}, t_i, t_{i+1}$

$$y(t) = P(t) + \frac{y^{(4)}(\xi(t))}{24}(t - t_{i-2})(t - t_{i-1})(t - t_i)(t - t_{i+1})$$

$$\frac{P(t_{i+1}) + \frac{3}{2}P(t_i) - 3P(t_{i-1}) - \frac{1}{2}P(t_{i-2})}{h} - 3f(t_i, y(t_i))]$$

$$f(t_i, y(t_i)) = \frac{d}{dt}y(t_i) = \frac{d}{dt}\left(P(t) + \frac{y^{(4)}(\xi(t))}{24}(t - t_{i-2})(t - t_{i-1})(t - t_i)(t - t_{i+1})\right]\Bigg|_{t = t_i}$$

$$= P'(t_i) + \frac{y^{(4)}(\xi(t_i))(2h)(h)(-h)}{24} = P'(t_i) - \frac{h^3 y^{(4)}(\xi(t_i))}{12}$$

$$= \underbrace{\frac{P(t_{i+1}) + \frac{3}{2}P(t_i) - 3P(t_{i-1}) - \frac{1}{2}P(t_{i-2})}{h} - 3P'(t_i)}_{=0} + \underbrace{\frac{h^3 y^{(4)}(\xi(t_i))}{4}} = \tau$$

$P = a + bt + ct^2 + dt^3$